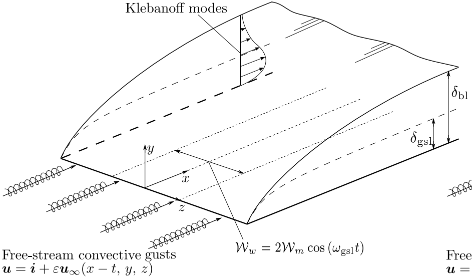
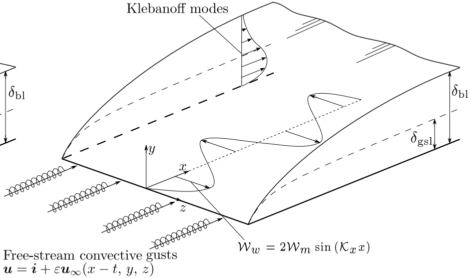
A comparison of the laminar streaks above a spanwise oscillating plate and a plate with spanwise wall forcing

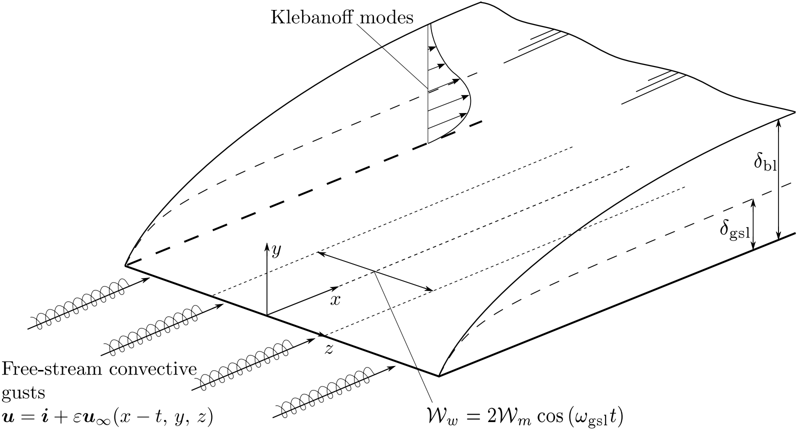




Peter Hicks
University of Aberdeen
p.hicks@abdn.ac.uk

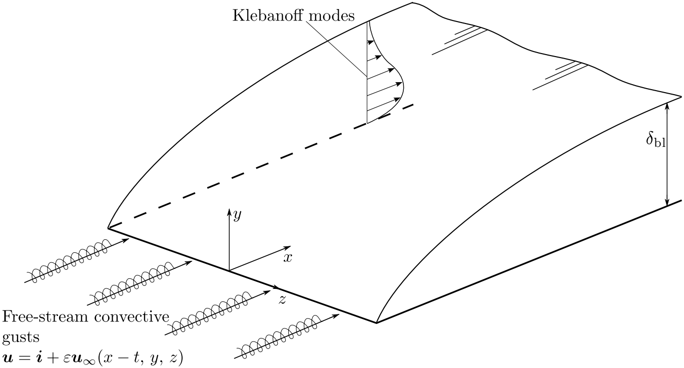
Pierre Ricco
University of Sheffield
p.ricco@sheffield.ac.uk

Laminar streak growth above an oscillating flat plate



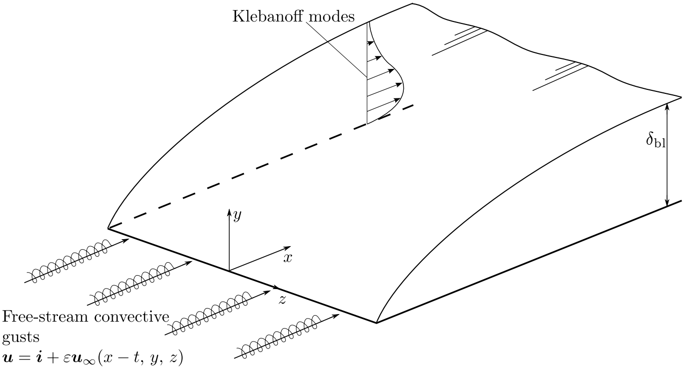
- Following the formulation of Leib, Wudrow and Goldstein (JFM, 1999, 380, pp. 169–203), disturbances in the boundary layer are coupled to disturbances in the free stream.
- Strouhal number measuring ratio of plate oscillations to free-stream gust oscillations: $N = \dfrac{\omega_{\mathsf{gsl}}}{k_x}$, where k_x is the streamwise frequency of the gust.

Free-stream turbulence (I)



- There are various forms of free-stream disturbances including vortical, acoustic and entropic fluctuations.
- This talk deals with incompressible flow, and will consider only vortical disturbances.

Free-stream turbulence (II)



- These can be expressed as

$$\boldsymbol{u} = \hat{\boldsymbol{\imath}} + \varepsilon \hat{\boldsymbol{u}}_\infty e^{\mathrm{i}(k_x x + k_y y + k_z z - k_x t)} + \text{c.c.},$$

  where we've scaled by the free-stream velocity and $\varepsilon \ll 1$ is a measure of the turbulence intensity in the free stream.
- Experiments have shown that it is low-frequency (long-wavelength) disturbances with $k_x \ll k_y$ and $k_x \ll k_z$, that penetrate the boundary layer most effectively.

Laminar streak growth above an oscillating flat plate

We now look for small perturbations about both the base flow, of the form

$$
\begin{aligned}
U =& U_{\mathsf{bl}}(x,\,y) + r_t\,u(x,\,y,\,z,\,t)\,,\\
V =& V_{\mathsf{bl}}(x,\,y) + r_t\,v(x,\,y,\,z,\,t)\,,\\
\mathcal{W} =& \mathcal{W}_{\mathsf{gsl}}(x,\,y,\,t) + r_t\,w(x,\,y,\,z,\,t)\,,\\
P =& -\frac{1}{2} + r_t\,p(x,\,y,\,z,\,t)\,,
\end{aligned}
$$

where the turbulent Reynolds number $r_t = \varepsilon R_\lambda \ll 1$, where $R_\lambda = U^*_\infty \lambda^*_z/\nu^*_\infty$.

Laminar streak growth above an oscillating flat plate

We now look for small perturbations about both the base flow, of the form

$$\begin{aligned} U =& U_{\mathsf{bl}}(x,\, y) + r_t\, u(x,\, y,\, z,\, t)\,, \\ V =& V_{\mathsf{bl}}(x,\, y) + r_t\, v(x,\, y,\, z,\, t)\,, \\ \mathcal{W} =& \mathcal{W}_{\mathsf{gsl}}(x,\, y,\, t) + r_t\, w(x,\, y,\, z,\, t)\,, \\ P =& -\frac{1}{2} + r_t\, p(x,\, y,\, z,\, t)\,, \end{aligned}$$

where the turbulent Reynolds number $r_t = \varepsilon R_\lambda \ll 1$, where $R_\lambda = U^*_\infty \lambda^*_z / \nu^*_\infty$.

Collecting terms that are O(1) implies:

- The streamwise and wall-normal base flow momentum equations uncouple from the spanwise base flow momentum equation.
- The streamwise and wall-normal velocity profiles match the usual Blasius boundary layer.

Laminar streak growth above an oscillating flat plate

We now look for small perturbations about both the base flow, of the form

$$
\begin{aligned}
U &= U_{\mathsf{bl}}(x,\,y) + r_t\,u(x,\,y,\,z,\,t)\,,\\
V &= V_{\mathsf{bl}}(x,\,y) + r_t\,v(x,\,y,\,z,\,t)\,,\\
\mathcal{W} &= \mathcal{W}_{\mathsf{gsl}}(x,\,y,\,t) + r_t\,w(x,\,y,\,z,\,t)\,,\\
P &= -\frac{1}{2} + r_t\,p(x,\,y,\,z,\,t)\,,
\end{aligned}
$$

where the turbulent Reynolds number $r_t = \varepsilon R_\lambda \ll 1$, where $R_\lambda = U^*_\infty \lambda^*_z / \nu^*_\infty$.

Collecting terms that are $\mathrm{O}(1)$ implies:

- The streamwise and wall-normal base flow momentum equations uncouple from the spanwise base flow momentum equation.
- The streamwise and wall-normal velocity profiles match the usual Blasius boundary layer.
- The spanwise base flow is given by

$$
\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial t} + U_{\mathsf{bl}}\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial x} + V_{\mathsf{bl}}\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial y} = \frac{1}{R_\lambda}\left(\frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial x^2} + \frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial y^2}\right),
$$

  subject to an oscillating plate ($y = 0,\ x > 0$), which has equation of motion

$$
(U_{\mathsf{bl}},\,V_{\mathsf{bl}},\,\mathcal{W}_{\mathsf{gsl}}) = (0,\,0,\,2\mathcal{W}_m\cos(\omega_{\mathsf{gsl}}t)) = \left(0,\,0,\,\mathcal{W}_m e^{\mathrm{i}\omega_{\mathsf{gsl}}t} + \mathcal{W}_m e^{-\mathrm{i}\omega_{\mathsf{gsl}}t}\right).
$$

The spanwise base flow

- Let's consider the spanwise base flow equation in more detail:

$$\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial t} + U_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial x} + V_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial y} = \frac{1}{R_\lambda} \left(\frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial x^2} + \frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial y^2} \right).$$

The spanwise base flow: relationship to a classical Stokes layer

- Let's consider the spanwise base flow equation in more detail:

$$\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial t} + U_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial x} + V_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial y} = \frac{1}{R_\lambda} \left(\frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial x^2} + \frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial y^2} \right).$$

- If we had a genuinely parallel boundary layer flow (i.e. $V_{\mathsf{bl}} = 0$ and $\mathcal{W}_{\mathsf{gsl}}$ is independent of x), then the spanwise base flow satisfies

$$\frac{\partial \mathcal{W}_{\mathsf{csl}}}{\partial t} = \frac{1}{R_\lambda} \frac{\partial^2 \mathcal{W}_{\mathsf{csl}}}{\partial y^2}.$$

The spanwise base flow: relationship to a classical Stokes layer

- Let's consider the spanwise base flow equation in more detail:

$$\frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial t} + U_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial x} + V_{\mathsf{bl}} \frac{\partial \mathcal{W}_{\mathsf{gsl}}}{\partial y} = \frac{1}{R_\lambda} \left(\frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial x^2} + \frac{\partial^2 \mathcal{W}_{\mathsf{gsl}}}{\partial y^2} \right).$$

- If we had a genuinely parallel boundary layer flow (i.e. $V_{\mathsf{bl}} = 0$ and $\mathcal{W}_{\mathsf{gsl}}$ is independent of x), then the spanwise base flow satisfies

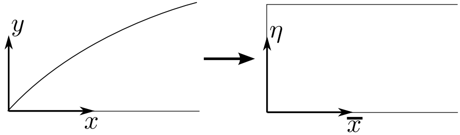
$$\frac{\partial \mathcal{W}_{\mathsf{csl}}}{\partial t} = \frac{1}{R_\lambda} \frac{\partial^2 \mathcal{W}_{\mathsf{csl}}}{\partial y^2}.$$

- With the oscillating plate boundary condition given previously, this is Stokes second problem, which has solution

$$\mathcal{W}_{\mathsf{csl}} = 2\mathcal{W}_m \exp\left(-\sqrt{\frac{\omega_{\mathsf{gsl}} R_\lambda}{2}} y \right) \cos\left(\omega_{\mathsf{gsl}} t - \sqrt{\frac{\omega_{\mathsf{gsl}} R_\lambda}{2}} y \right).$$

Boundary layer coordinate transformation and asymptotics

We look at the distinguished limit $k_x R_\lambda = \mathrm{O}(1)$ for $R_\lambda \gg 1$.



In this limit, we make the coordinate transformation

$$\{x,\, y,\, z,\, t\} \to \left\{\overline{x},\, \eta,\, z,\, \overline{t}\right\},$$

where

$$\overline{x} = k_x x = \mathrm{O}(1)\,, \qquad \eta = y\left(\frac{R_\lambda}{2x}\right)^{\frac{1}{2}} = y\left(\frac{k_x R_\lambda}{2\overline{x}}\right)^{\frac{1}{2}}, \qquad \overline{t} = k_x t.$$

With this change of coordinates we've moved far enough downstream that streamwise momentum diffusion terms and the streamwise pressure gradient (in the disturbance momentum equations), are negligible.

The spanwise base flow: a generalized Stokes layer

- In the same limit and coordinate transfer, the generalized spanwise base flow is periodic in $\bar{t}$, and hence we look for a solution of the form

$$\mathcal{W}_{\mathsf{gsl}}\left(\overline{x},\,\eta,\,\bar{t}\right)=\frac{k_x}{k_z}W_{\mathsf{gsl}}\left(\overline{x},\,\eta,\,\bar{t}\right)=\frac{k_x}{k_z}W\left(\overline{x},\,\eta\right)e^{\mathrm{i}N\bar{t}}+\frac{k_x}{k_z}W^{\star}\left(\overline{x},\,\eta\right)e^{-\mathrm{i}N\bar{t}},$$

where $\star$ denotes a complex conjugate.

The spanwise base flow: a generalized Stokes layer

- In the same limit and coordinate transfer, the generalized spanwise base flow is periodic in $\overline{t}$, and hence we look for a solution of the form

$$\mathcal{W}_{\mathsf{gsl}}\left(\overline{x},\, \eta,\, \overline{t}\right) = \frac{k_x}{k_z} W_{\mathsf{gsl}}\left(\overline{x},\, \eta,\, \overline{t}\right) = \frac{k_x}{k_z} W(\overline{x},\, \eta)\, e^{\mathrm{i}N\overline{t}} + \frac{k_x}{k_z} W^{\star}(\overline{x},\, \eta)\, e^{-\mathrm{i}N\overline{t}},$$

  where $\star$ denotes a complex conjugate.

- The spanwise base flow is thus given by (the parabolic PDE)

$$\mathrm{i}NW + F' \frac{\partial W}{\partial \overline{x}} - \frac{F}{2\overline{x}} \frac{\partial W}{\partial \eta} = \frac{1}{2\overline{x}} \frac{\partial^2 W}{\partial \eta^2},$$

  subject to:

  - large-η conditions:

$$W \to 0 \quad \text{as} \quad \eta \to \infty;$$

  - on the plate:

$$W(\overline{x},\, 0) = \frac{k_z}{k_x} \mathcal{W}_m = W_m \quad \text{for} \quad \overline{x} > 0;$$

  - initial conditions (obtained for $\overline{x} \ll 1$):

$$W \sim W_m \left(1 - F'\right).$$

WKBJ structure for large $\hat{x} = N\overline{x}$ generalized Stokes layers

- To study the spanwise base flow behaviour for $\hat{x} = N\overline{x} \gg 1$, we have the equation

$$\mathrm{i}W + F' \frac{\partial W}{\partial \hat{x}} - \frac{F}{2\hat{x}} \frac{\partial W}{\partial \eta} = \frac{1}{2\hat{x}} \frac{\partial^2 W}{\partial \eta^2}.$$

WKBJ structure for large $\hat{x} = N\overline{x}$ generalized Stokes layers

- To study the spanwise base flow behaviour for $\hat{x} = N\overline{x} \gg 1$, we have the equation

$$\mathrm{i}W + F'\frac{\partial W}{\partial \hat{x}} - \frac{F}{2\hat{x}}\frac{\partial W}{\partial \eta} = \frac{1}{2\hat{x}}\frac{\partial^2 W}{\partial \eta^2}.$$

- Note $\hat{x} = N\overline{x} \gg 1$ can be achieved either by:
  1. going a large distance downstream,
  2. by having rapid plate oscillations.

WKBJ structure for large $\hat{x} = N\overline{x}$ generalized Stokes layers

- To study the spanwise base flow behaviour for $\hat{x} = N\overline{x} \gg 1$, we have the equation

$$\mathrm{i}W + F'\frac{\partial W}{\partial \hat{x}} - \frac{F}{2\hat{x}}\frac{\partial W}{\partial \eta} = \frac{1}{2\hat{x}}\frac{\partial^2 W}{\partial \eta^2}.$$

- Note $\hat{x} = N\overline{x} \gg 1$ can be achieved either by:
  1. going a large distance downstream,
  2. by having rapid plate oscillations.

- If we look for a large $\hat{x}$ WKBJ solution of the form

$$W = \overline{W}(\hat{x}, \eta)\, e^{-(2\hat{x})^{1/2}\Theta(\eta)},$$

then

$$\Theta(\eta) = \frac{(1+\mathrm{i})\eta}{\sqrt{2}}, \quad \text{and} \quad \overline{W} = \frac{W_m F''(\eta)}{F''(0)} \exp\left(\frac{F\eta}{2}\right).$$

WKBJ structure for large $\hat{x} = N\overline{x}$ generalized Stokes layers

- To study the spanwise base flow behaviour for $\hat{x} = N\overline{x} \gg 1$, we have the equation

$$\mathrm{i}W + F' \frac{\partial W}{\partial \hat{x}} - \frac{F}{2\hat{x}} \frac{\partial W}{\partial \eta} = \frac{1}{2\hat{x}} \frac{\partial^2 W}{\partial \eta^2}.$$

- Note $\hat{x} = N\overline{x} \gg 1$ can be achieved either by:
  1. going a large distance downstream,
  2. by having rapid plate oscillations.

- If we look for a large $\hat{x}$ WKBJ solution of the form

$$W = \overline{W}(\hat{x}, \eta)\, e^{-(2\hat{x})^{1/2}\Theta(\eta)},$$

  then

$$\Theta(\eta) = \frac{(1+\mathrm{i})\eta}{\sqrt{2}}, \quad \text{and} \quad \overline{W} = \frac{W_m F''(\eta)}{F''(0)} \exp\left(\frac{F\eta}{2}\right).$$

- Consequently if we return to dimensional variables, then

$$W_{\text{gsl}}\big|_{\hat{x} \gg 1} \sim W_m \exp\left(-(1+\mathrm{i})\sqrt{\frac{\omega^*_{\text{gsl}}}{2\nu^*}}\, y^*\right) \exp\left(\mathrm{i}\omega^*_{\text{gsl}} t^*\right), \quad \text{for} \quad \hat{x} \gg 1,$$

  i.e. we get a true Stokes layer for $\hat{x} \gg 1$.

Spanwise base flow with steady wall forcing

- The spanwise flow with steady spanwise forcing satisfies (the parabolic PDE)

$$F' \frac{\partial W}{\partial \overline{x}} - \frac{F}{2\overline{x}} \frac{\partial W}{\partial \eta} = \frac{1}{2\overline{x}} \frac{\partial^2 W}{\partial \eta^2},$$

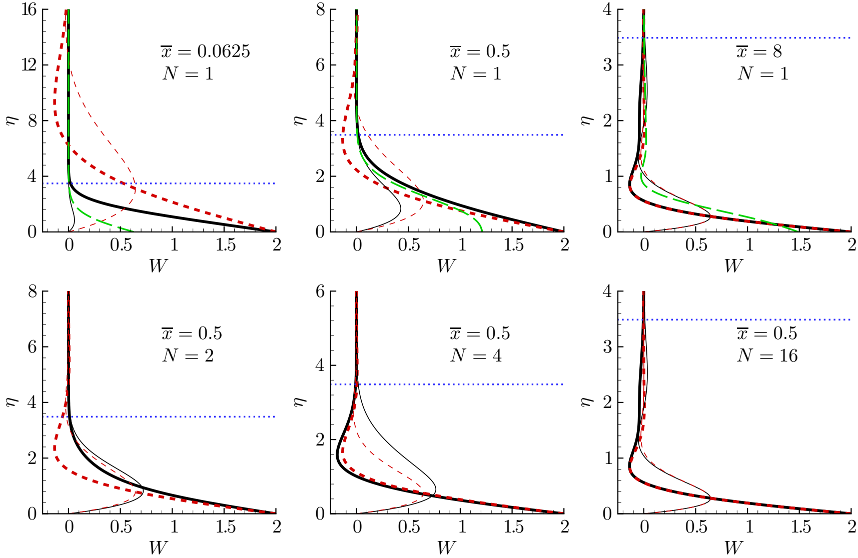
subject to the boundary conditions at the plate

$$W(x,\, 0) = 2W_m \sin(\mathcal{K}_x \overline{x})\,.$$

- This equation differs from the spanwise base flow equation for the oscillating plate through the absence of the time dependent term.

- More detailed information about the streak evolution with steady forcing is available in Ricco (2011, Phys. Fluids).

Spanwise base flow: comparison with Stokes 2nd solution



GSL, CSL, steady forcing for $\mathcal{K}_x = 5$, and Blasius layer thickness $0.99U^*_\infty$.

Disturbance equation formulation

Disturbance equations: initial value problem

Collecting terms at $\mathrm{O}(r_t)$ gives the linearized disturbance equations:

$$\frac{\partial \overline{u}}{\partial \overline{x}} - \frac{\eta}{2\overline{x}}\frac{\partial \overline{u}}{\partial \eta} + \frac{\partial \overline{v}}{\partial \eta} + \frac{1}{k_z}\frac{\partial \overline{w}}{\partial z} = 0,$$

$$\frac{\partial \overline{u}}{\partial \bar{t}} + F'\frac{\partial \overline{u}}{\partial \overline{x}} - \frac{\eta F''}{2\overline{x}}\overline{u} - \frac{F}{2\overline{x}}\frac{\partial \overline{u}}{\partial \eta} + F''\overline{v} + \frac{W_{\text{gsl}}}{k_z}\frac{\partial \overline{u}}{\partial z} = \frac{1}{2\overline{x}}\frac{\partial^2 \overline{u}}{\partial \eta^2} + \frac{1}{k_x R_\lambda}\frac{\partial^2 \overline{u}}{\partial z^2},$$

$$\frac{\partial \overline{v}}{\partial \bar{t}} + \frac{(\eta F')'}{2\overline{x}}\overline{v} + F'\frac{\partial \overline{v}}{\partial \overline{x}} - \frac{F}{2\overline{x}}\frac{\partial \overline{v}}{\partial \eta} + \frac{\left[F - \eta\left(\eta F'\right)'\right]}{(2\overline{x})^2}\overline{u} + \frac{W_{\text{gsl}}}{k_z}\frac{\partial \overline{v}}{\partial z} = -\frac{1}{2\overline{x}}\frac{\partial \overline{p}}{\partial \eta} + \frac{1}{2\overline{x}}\frac{\partial^2 \overline{v}}{\partial \eta^2} + \frac{1}{k_x R_\lambda}\frac{\partial^2 \overline{v}}{\partial z^2},$$

$$\frac{\partial \overline{w}}{\partial \bar{t}} + F'\frac{\partial \overline{w}}{\partial \overline{x}} + \left(\frac{\partial W_{\text{gsl}}}{\partial \overline{x}} - \frac{\eta}{2\overline{x}}\frac{\partial W_{\text{gsl}}}{\partial \eta}\right)\overline{u} - \frac{F}{2\overline{x}}\frac{\partial \overline{w}}{\partial \eta} + \frac{\partial W_{\text{gsl}}}{\partial \eta}\overline{v} + \frac{W_{\text{gsl}}}{k_z}\frac{\partial \overline{w}}{\partial z} = -\frac{k_z}{k_x R_\lambda}\frac{\partial \overline{p}}{\partial z} + \frac{1}{2\overline{x}}\frac{\partial^2 \overline{w}}{\partial \eta^2} + \frac{1}{k_x R_\lambda}\frac{\partial^2 \overline{w}}{\partial z^2}.$$

Here the terms in red are the additional terms resulting from the spanwise base flow, which are not present in the stationary plate case.

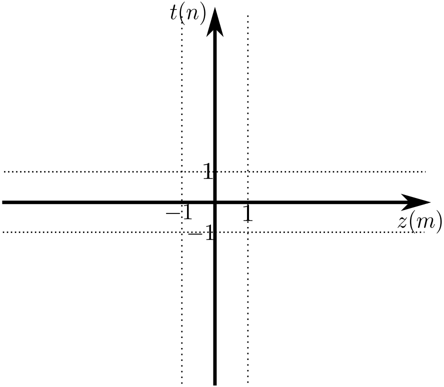
The base flow and disturbance behaviour is periodic in z and $\bar{t}$, so we look for a Fourier series expansion in these variables.

- Consider two-dimensional Fourier series for the velocity components and the pressure of the form

$$\overline{\alpha} = \sum_{m,n=-\infty}^{\infty} \overline{\alpha}^{[n,m]}(\overline{x},\, \eta)\, e^{\mathrm{i}mk_z z + \mathrm{i}n\overline{t}},$$

where $\overline{\alpha}$ is one of $\overline{u}$, $\overline{v}$, $\overline{w}$ or $\overline{p}$.

Map of Fourier coefficients:



Which terms in the two-dimensional Fourier series are non-zero? (I)

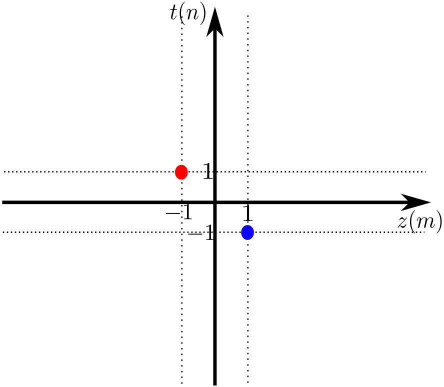
- Consider two-dimensional Fourier series for the velocity components and the pressure of the form

$$\overline{\alpha} = \sum_{m,n=-\infty}^{\infty} \overline{\alpha}^{[n,m]}(\overline{x},\, \eta)\, e^{\mathrm{i}mk_z z + \mathrm{i}n\overline{t}},$$

  where $\overline{\alpha}$ is one of $\overline{u}$, $\overline{v}$, $\overline{w}$ or $\overline{p}$.

- The terms which directly match the free-stream forcing are $\overline{\alpha}^{[-1,1]} e^{\mathrm{i}k_z z - \mathrm{i}\overline{t}}$ and $\overline{\alpha}^{[1,-1]} e^{-\mathrm{i}k_z z + \mathrm{i}\overline{t}}$, so these terms are definitely non-zero.

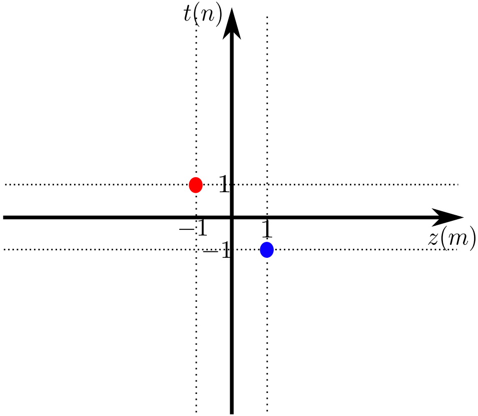
Map of Fourier coefficients:



Which terms in the two-dimensional Fourier series are non-zero? (II)

- In a linear theory we have terms like $W\overline{\alpha}$, where $W = We^{\mathrm{i}N\bar{t}} + W^{\star}e^{-\mathrm{i}N\bar{t}}$ is the oscillating spanwise base flow.

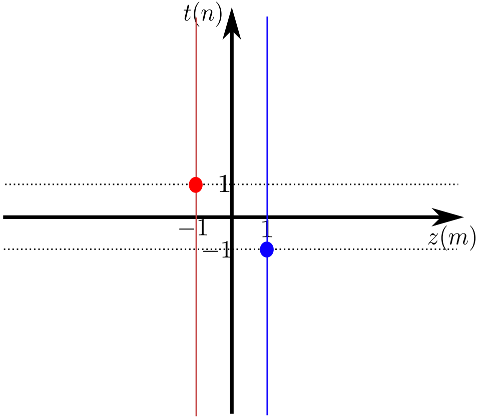
Map of Fourier coefficients:



- In a linear theory we have terms like $W\overline{\alpha}$, where $W = We^{\mathrm{i}N\bar{t}} + W^{\star}e^{-\mathrm{i}N\bar{t}}$ is the oscillating spanwise base flow.

- This generates terms of the form $e^{\mathrm{i}mk_z z+\mathrm{i}(n\pm N)\bar{t}}$, so terms involving the spanwise base flow act to shift the Fourier series in time and couple Fourier coefficients.

Map of Fourier coefficients:

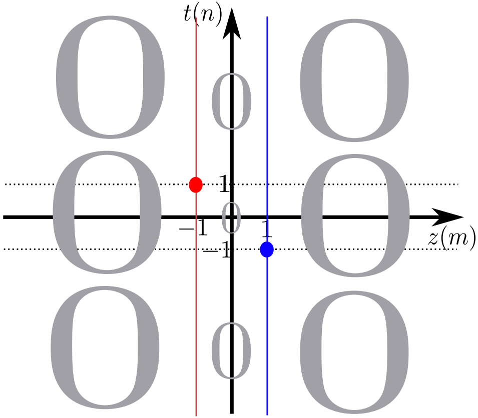


- In a linear theory we have terms like $W\overline{\alpha}$, where $W = We^{\mathrm{i}N\bar{t}} + W^{\star}e^{-\mathrm{i}N\bar{t}}$ is the oscillating spanwise base flow.

- This generates terms of the form $e^{\mathrm{i}mk_z z+\mathrm{i}(n\pm N)\bar{t}}$, so terms involving the spanwise base flow act to shift the Fourier series in time and couple Fourier coefficients.

- Therefore if $m \neq \pm 1$, then these Fourier nodes are not forced, and the spanwise base flow does not couple these modes to non-zero modes, and hence all these modes are zero.

Map of Fourier coefficients:

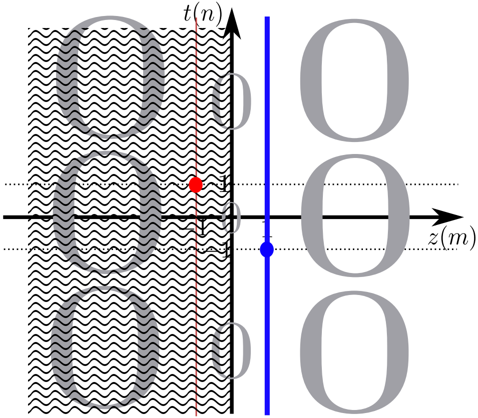


- Finally the velocity components and the pressure are real valued, so the conjugation conditions imply:

$$\begin{aligned}
\overline{u}^{[-n,-m]} &= \overline{u}^{[n,m]\star}, \\
\overline{v}^{[-n,-m]} &= \overline{v}^{[n,m]\star}, \\
\overline{w}^{[-n,-m]} &= \overline{w}^{[n,m]\star}, \\
\overline{p}^{[-n,-m]} &= \overline{p}^{[n,m]\star}.
\end{aligned}$$

- **Therefore if we can compute one family of Fourier coefficients (either with $m=1$ or $m=-1$), then we can reconstruct the full solution.**

Map of Fourier coefficients:



Boundary layer variable expansion

Next we'll again expand the disturbance velocities and pressure as:

$$u\big(\overline{x},\,\eta,\,z,\,\overline{t}\big) = Q\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{u}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$v\big(\overline{x},\,\eta,\,z,\,\overline{t}\big) = Q\left(\frac{2\overline{x}k_x}{R_\lambda}\right)^{1/2}\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{v}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$w\big(\overline{x},\,\eta,\,z,\,\overline{t}\big) = Q\sum_{n=-\infty}^{\infty}\overline{w}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$p\big(\overline{x},\,\eta,\,z,\,\overline{t}\big) = Q\kappa_z\left(\frac{k_x}{R_\lambda}\right)^{1/2}\sum_{n=-\infty}^{\infty}\overline{p}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

where we've used scaled wave numbers: $\kappa_z = k_z/(k_x R_\lambda)^{1/2}$ and $\kappa_y = k_y/(k_x R_\lambda)^{1/2}$.

Boundary layer variable expansion

Next we'll again expand the disturbance velocities and pressure as:

$$u(\overline{x},\,\eta,\,z,\,\overline{t}) = Q\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{u}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$v(\overline{x},\,\eta,\,z,\,\overline{t}) = Q\left(\frac{2\overline{x}k_x}{R_\lambda}\right)^{1/2}\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{v}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$w(\overline{x},\,\eta,\,z,\,\overline{t}) = Q\sum_{n=-\infty}^{\infty}\overline{w}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

$$p(\overline{x},\,\eta,\,z,\,\overline{t}) = Q\kappa_z\left(\frac{k_x}{R_\lambda}\right)^{1/2}\sum_{n=-\infty}^{\infty}\overline{p}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},$$

where we've used scaled wave numbers: $\kappa_z = k_z/(k_x R_\lambda)^{1/2}$ and $\kappa_y = k_y/(k_x R_\lambda)^{1/2}$.

- The factor Q guarantees the a free-stream vortical disturbance in the base flow region satisfies the no-penetration condition at the wall;

Boundary layer variable expansion

Next we'll again expand the disturbance velocities and pressure as:

$$u(\overline{x},\, \eta,\, z,\, \overline{t}) = Q\,\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty} \overline{u}^{[n]}(\overline{x},\, \eta)\, e^{\mathrm{i}k_z z + \mathrm{i}n\overline{t}},$$

$$v(\overline{x},\, \eta,\, z,\, \overline{t}) = Q\left(\frac{2\overline{x}k_x}{R_\lambda}\right)^{1/2}\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty} \overline{v}^{[n]}(\overline{x},\, \eta)\, e^{\mathrm{i}k_z z + \mathrm{i}n\overline{t}},$$

$$w(\overline{x},\, \eta,\, z,\, \overline{t}) = Q\sum_{n=-\infty}^{\infty} \overline{w}^{[n]}(\overline{x},\, \eta)\, e^{\mathrm{i}k_z z + \mathrm{i}n\overline{t}},$$

$$p(\overline{x},\, \eta,\, z,\, \overline{t}) = Q\kappa_z\left(\frac{k_x}{R_\lambda}\right)^{1/2}\sum_{n=-\infty}^{\infty} \overline{p}^{[n]}(\overline{x},\, \eta)\, e^{\mathrm{i}k_z z + \mathrm{i}n\overline{t}},$$

where we've used scaled wave numbers: $\kappa_z = k_z/(k_x R_\lambda)^{1/2}$ and $\kappa_y = k_y/(k_x R_\lambda)^{1/2}$.

- The factor Q guarantees the a free-stream vortical disturbance in the base flow region satisfies the no-penetration condition at the wall;
- The other scaling factors guarantee that $\overline{u}^{[-1]}$, $\overline{v}^{[-1]}$, $\overline{w}^{[-1]}$ and $\overline{p}^{[-1]}$ are all the same size.

Boundary layer variable expansion

Next we'll again expand the disturbance velocities and pressure as:

$$
\begin{aligned}
u(\overline{x},\,\eta,\,z,\,\overline{t}) &= Q\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{u}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},\\
v(\overline{x},\,\eta,\,z,\,\overline{t}) &= Q\left(\frac{2\overline{x}k_x}{R_\lambda}\right)^{1/2}\frac{k_z}{k_x}\sum_{n=-\infty}^{\infty}\overline{v}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},\\
w(\overline{x},\,\eta,\,z,\,\overline{t}) &= Q\sum_{n=-\infty}^{\infty}\overline{w}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},\\
p(\overline{x},\,\eta,\,z,\,\overline{t}) &= Q\kappa_z\left(\frac{k_x}{R_\lambda}\right)^{1/2}\sum_{n=-\infty}^{\infty}\overline{p}^{[n]}(\overline{x},\,\eta)\,e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}},
\end{aligned}
$$

where we've used scaled wave numbers: $\kappa_z = k_z/(k_x R_\lambda)^{1/2}$ and $\kappa_y = k_y/(k_x R_\lambda)^{1/2}$.

- The factor Q guarantees the a free-stream vortical disturbance in the base flow region satisfies the no-penetration condition at the wall;
- The other scaling factors guarantee that $\overline{u}^{[-1]}$, $\overline{v}^{[-1]}$, $\overline{w}^{[-1]}$ and $\overline{p}^{[-1]}$ are all the same size.
- This time $\overline{u}^{[n]}(\overline{x},\,\eta)$, $\overline{v}^{[n]}(\overline{x},\,\eta)$, $\overline{w}^{[n]}(\overline{x},\,\eta)$ and $\overline{p}^{[n]}(\overline{x},\,\eta)$ are the Fourier coefficients multipling $e^{\mathrm{i}k_z z+\mathrm{i}n\overline{t}}$.

The LUBR equations for flow above an oscillating plate

For a scaled spanwise baseflow $W = We^{\mathrm{i}N\bar{t}} + W^{\star}e^{-\mathrm{i}N\bar{t}}$, with $N \in \mathbb{N}$, we can collect up the coefficients, we find for each n that:

$$\frac{\partial \overline{u}^{[n]}}{\partial \overline{x}} - \frac{\eta}{2\overline{x}}\frac{\partial \overline{u}^{[n]}}{\partial \eta} + \frac{\partial \overline{v}^{[n]}}{\partial \eta} + \mathrm{i}\overline{w}^{[n]} = 0,$$

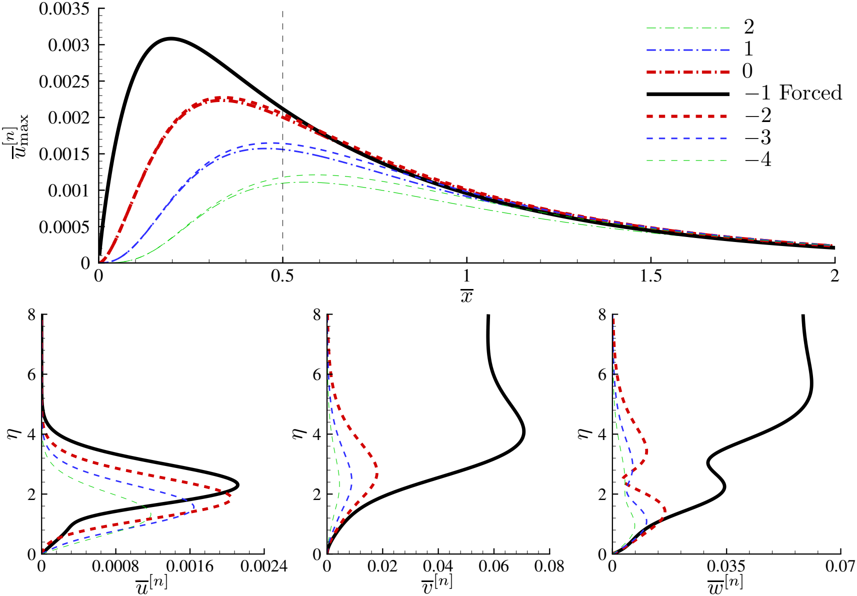
$$\left(\mathrm{i}n + \kappa_z^2 - \frac{\eta F''}{2\overline{x}}\right)\overline{u}^{[n]} + F'\frac{\partial \overline{u}^{[n]}}{\partial \overline{x}} - \frac{F}{2\overline{x}}\frac{\partial \overline{u}^{[n]}}{\partial \eta} - \frac{1}{2\overline{x}}\frac{\partial^2 \overline{u}^{[n]}}{\partial \eta^2} + F''\overline{v}^{[n]} + \mathrm{i}W\overline{u}^{[n-N]} + \mathrm{i}W^{\star}\overline{u}^{[n+N]} = 0,$$

$$\left(\mathrm{i}n + \kappa_z^2 + \frac{(\eta F')'}{2\overline{x}}\right)\overline{v}^{[n]} + F'\frac{\partial \overline{v}^{[n]}}{\partial \overline{x}} - \frac{F}{2\overline{x}}\frac{\partial \overline{v}^{[n]}}{\partial \eta} - \frac{1}{2\overline{x}}\frac{\partial^2 \overline{v}^{[n]}}{\partial \eta^2} + \frac{\left(F - \eta\,(\eta F')'\right)}{(2\overline{x})^2}\overline{u}^{[n]} + \frac{1}{2\overline{x}}\frac{\partial \overline{p}^{[n]}}{\partial \eta} + \mathrm{i}W\overline{v}^{[n-N]} + \mathrm{i}W^{\star}\overline{v}^{[n+N]} = 0,$$

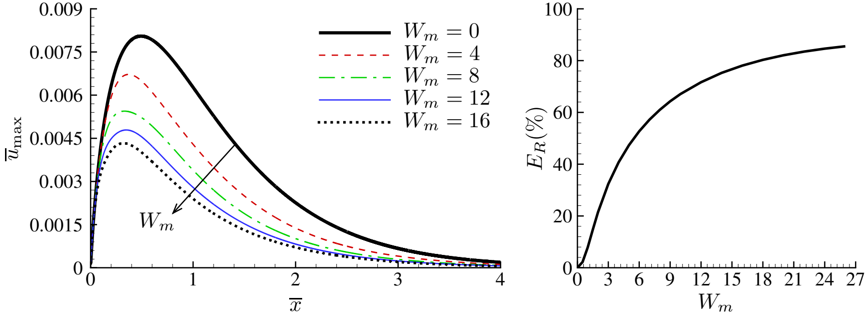
$$\left(\mathrm{i}n + \kappa_z^2\right)\overline{w}^{[n]} + F'\frac{\partial \overline{w}^{[n]}}{\partial \overline{x}} - \frac{F}{2\overline{x}}\frac{\partial \overline{w}^{[n]}}{\partial \eta} - \frac{1}{2\overline{x}}\frac{\partial^2 \overline{w}^{[n]}}{\partial \eta^2} + \mathrm{i}\kappa_z^2\overline{p}^{[n]} + \left(\frac{\partial W}{\partial \overline{x}} - \frac{\eta}{2\overline{x}}\frac{\partial W}{\partial \eta}\right)\overline{u}^{[n-N]} + \frac{\partial W}{\partial \eta}\overline{v}^{[n-N]} + \mathrm{i}W\overline{w}^{[n-N]} + \left(\frac{\partial W^{\star}}{\partial \overline{x}} - \frac{\eta}{2\overline{x}}\frac{\partial W^{\star}}{\partial \eta}\right)\overline{u}^{[n+N]} + \frac{\partial W^{\star}}{\partial \eta}\overline{v}^{[n+N]} + \mathrm{i}W^{\star}\overline{w}^{[n+N]} = 0.$$

Disturbance profile results

Streak evolution for $\kappa_z = \kappa_y = 1$, $N = 1$ and $W_m = 8$ plate oscillations

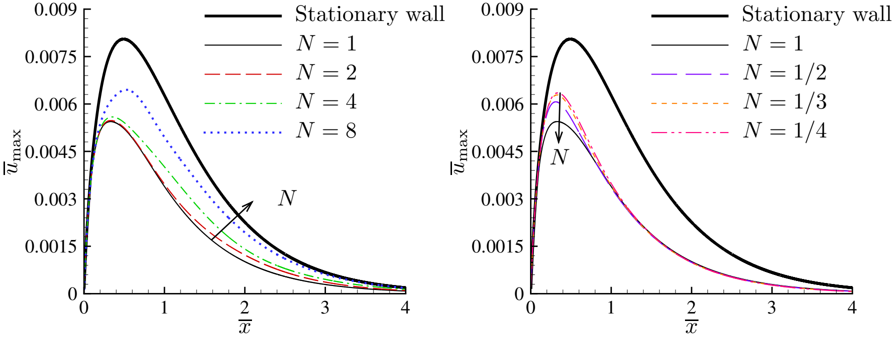


Energy reduction with plate oscillation amplitude ($\kappa_y = \kappa_z = 1$ and $N = 1$)

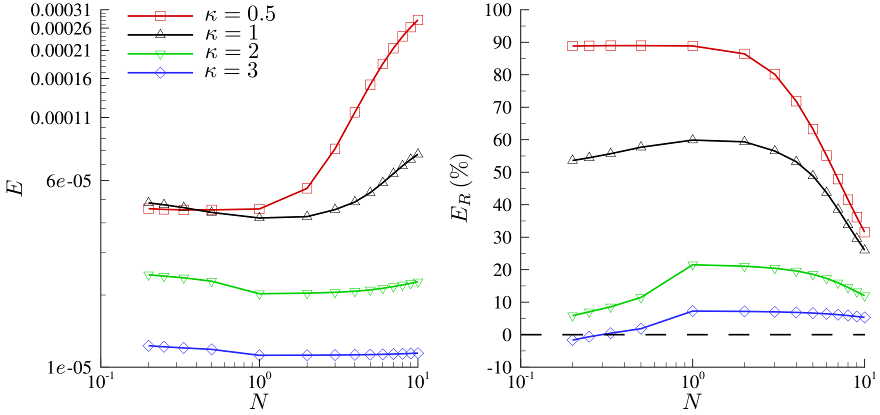


- Here $\overline{u}_{\max}(\overline{x}) = \max_{\eta} \{\overline{u}_{\text{rms}}(\overline{x}, \eta)\}$;
- The energy is obtained by integrating $|\overline{u}_{\text{rms}}|^2$ over both $\overline{x}$ and η;
- Alternative values of κ_y, κ_z and N can produce increases in the streak energy as W_m increases.

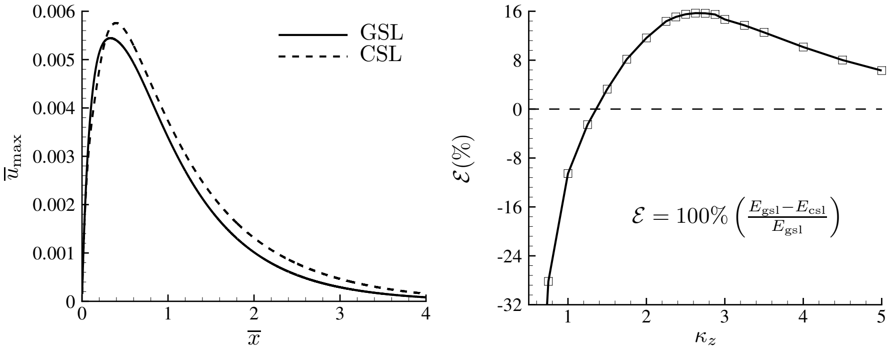
Energy reduction with plate oscillation frequency (I)



Energy reduction with plate oscillation frequency (II)

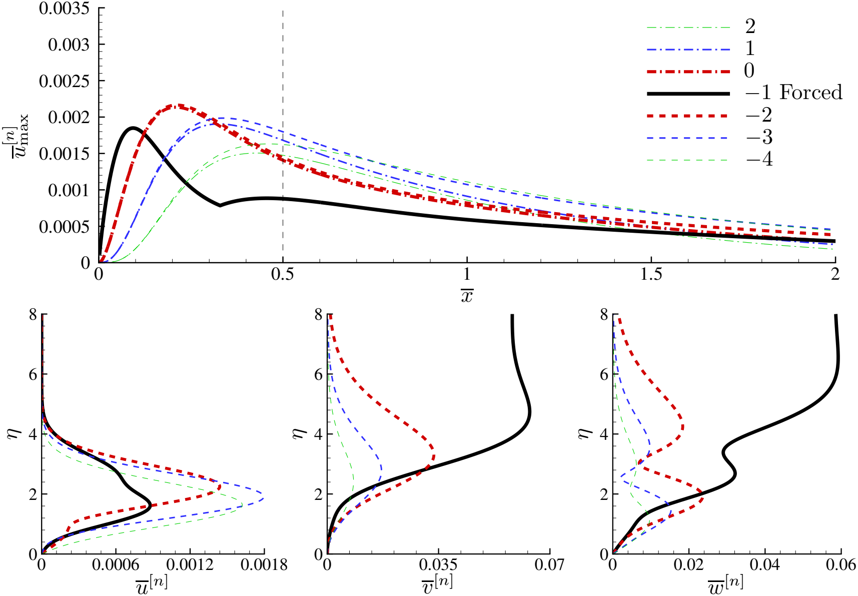


Energy comparison for streaks with CSL and GSL spanwise base flow

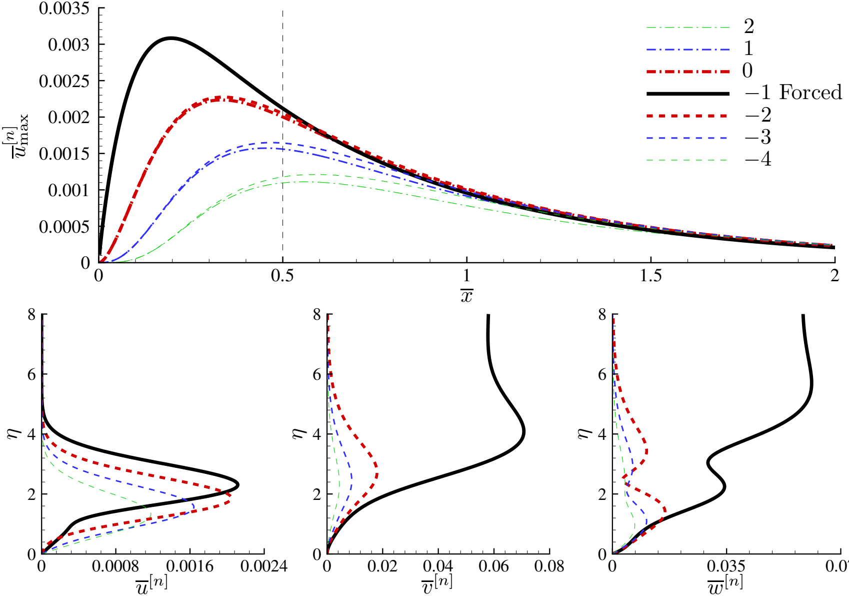


- Hack and Zaki (Phys. Fluids, 2012) looked at this problem with a CSL as the spanwise base flow;
- The actual energy contained in these two streaks is actually quite similar;
- However, we'll see that there are significant differences in the streak profiles;
- These differences can be increased further by altering the free-stream disturbance properties.

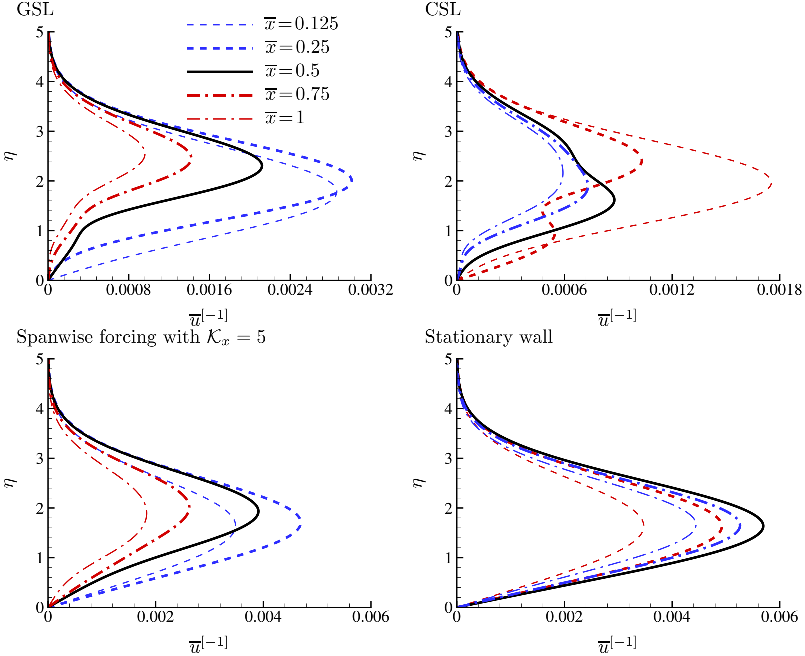
Comparison of **CSL** and GSL steamwise disturbance velocity evolution (I)



Comparison of CSL and **GSL** streamwise disturbance velocity evolution (II)



Comparison of wall normal velocity profiles



Conclusions and summary

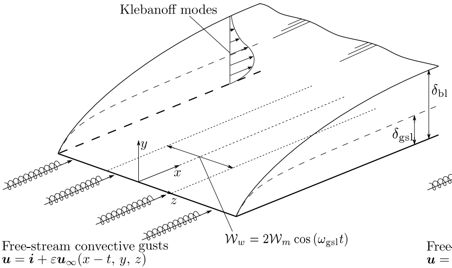
1. Spanwise plate oscillations and steady spanwise wall forcing can both reduced (and also increase), the energy contained in laminar streaks in a boundary layer depending on the properties of the free-stream disturbance (κ_z, κ_y), the amplitude of the oscillation (W_m), and either the frequency of the plate oscillation (ω_{gsl}) or the wavelength of the steady forcing ($\mathcal{K}_x$).

2. The non-parallel effects from the Blasius boundary layer are required to accurately describe the problem, as the laminar streaks generated in a generalized Stokes layer are markedly different to those arising when a classical Stokes layer is used.

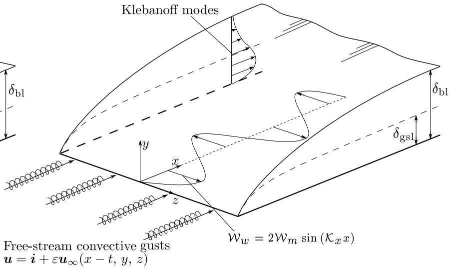
3. These results are contained in a J. Fluid Mech. paper currently under revision.

This work was partially supported by EPSRC First Grant EP/I033173/1.

Index



Klebanoff modes
δ_{bl}
δ_{gsl}
y
x
z
$\mathcal{W}_w = 2\mathcal{W}_m \cos(\omega_{\mathrm{gsl}} t)$
Free-stream convective gusts
$\boldsymbol{u} = \boldsymbol{i} + \varepsilon \boldsymbol{u}_\infty(x - t,\, y,\, z)$



Klebanoff modes
δ_{bl}
δ_{gsl}
y
x
z
$\mathcal{W}_w = 2\mathcal{W}_m \sin(\mathcal{K}_x x)$
Free-stream convective gusts
$\boldsymbol{u} = \boldsymbol{i} + \varepsilon \boldsymbol{u}_\infty(x - t,\, y,\, z)$