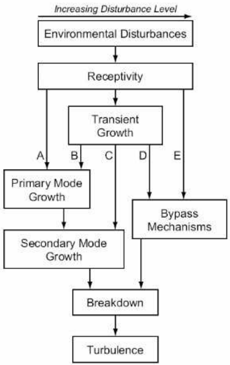
Nonlinear response of a compressible boundary-layer to free-stream vortical disturbances

Elena Marensi[1], Pierre Ricco[1], Xuesong Wu[2]

[1]Department of Mechanical Engineering, The University of Sheffield

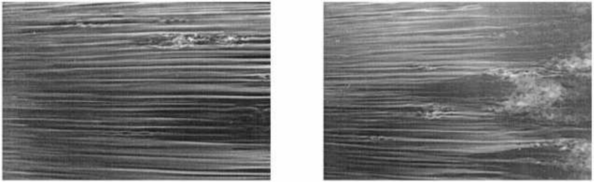
[2]Department of Mathematics, Imperial College London

- Free-stream turbulence (FST)
- Turbulence level $\rightarrow$ different paths to turbulence:
  - $Tu < 0.1\%$ orderly route through Tollmien-Schlichting waves.
  - $Tu > 1\%$ bypass transition through Klebanoff modes.

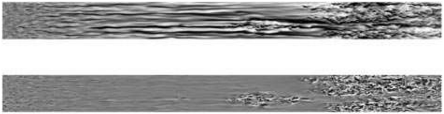


Source: *Morkovin (1984)*

Flow visualisations



Experiments. Source: *Matsubara & Alfredsson (2001)*



Direct Numerical Simulations. Source: *Jacobs & Durbin (2001)*

Theoretical works

- **Stewartson mode**:
  - Taylor (1939), Stewartson (1957).
  - Small modulation of boundary layer thickness $u \sim \eta F''$.
- **Optimal transient growth theory**:
  - Andersson *et al.* (1999), Luchini (2000).
  - Max energy growth $\rightarrow$ adjoint equations.
- **Orr-Sommerfeld spectrum theory**:
  - Jacobs & Durbin (2001), Zaki & Durbin (2005), Joo & Durbin (2012).
  - FST $\rightarrow$ continuous spectra of the O-S and Squire operators.
- **Boundary region approach**:
  - Leib *et al.* (1999), Ricco & Wu (2007), Ricco, Luo & Wu (2011).
  - Rigorous asymptotic theory $\rightarrow$ unsteady boundary region eqs.

Objectives

- Nonlinear evolution of unsteady streaks driven by medium-intensity free-stream vortical disturbances in a compressible boundary layer.
- First step for { secondary instability analysis; prediction and control of compressible bypass transition.

Framework

- Combination of the linear compressible and nonlinear incompressible analyses by Ricco & Wu (2007) and Ricco, Luo & Wu (2011).

Relevance

- Turbomachinery, wind tunnel (although acoustic modes are not considered) and flight conditions.

Scaling and assumptions

- Air flow: U^*_∞, T^*_∞ past a flat plate.
- Turbulent vortical fluctuations of the convected-gust type.
- FST: a pair of oblique vortical modes.

$$\mathbf{u} - \hat{\mathbf{i}} = \epsilon \mathbf{u}_\infty(x-t, y, z) = \epsilon(\hat{\mathbf{u}}^\infty_+ e^{ik_x(x-t)+ik_y y} + \hat{\mathbf{u}}^\infty_- e^{-ik_x(x-t)-ik_y y})e^{ik_z z} + c.c.$$

where $\epsilon \ll 1$ is a measure of the turbulence level.

- Reference length scale: $\lambda^* = 1/k^*_z$.
- Mach number $M = \mathcal{O}(1)$.
- Nonlinear effects $r_t = \epsilon R_\lambda = \mathcal{O}(1)$, with $R_\lambda = U^*_\infty \lambda^*/\nu^*_\infty \gg 1$.
- Scaled streamwise coordinate: $\bar{x} = k_x x = \mathcal{O}(1)$, with $k_x \ll 1$.
- Scaled spanwise wavenumber $\kappa \equiv \dfrac{k_z}{(k_x R_\lambda)^{1/2}} = \mathcal{O}\left(\dfrac{\delta^*}{\lambda^*}\right)$.

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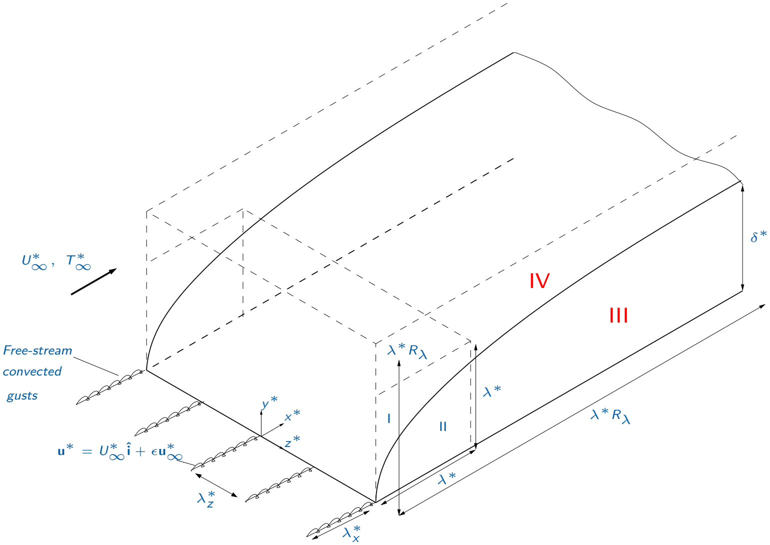

- *Region I*: Rapid Distortion Theory (Hunt, 1973 and Goldstein, 1978).
- *Region II*: Unsteady boundary layer eqs: $\kappa = \mathcal{O}\left(\delta^*/\lambda^*\right) \ll 1$.
- *Region III*: Unsteady boundary region eqs: $\kappa = \mathcal{O}\left(\delta^*/\lambda^*\right) = \mathcal{O}(1)$.
  - ~~$\partial^2/\partial\bar{x}^2$~~ and ~~$\partial p/\partial\bar{x}$~~
  - Elliptic in z and parabolic in $\bar{x}$
- *Region IV*: Fully nonlinear flow due to displacement effect.

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The inner region

- Blasius layer and perturbation induced by free-stream turbulence

$$\{u, v, w, p, t\} = \left\{U, V, 0, -\frac{1}{2}, T\right\}(\bar{x}, \eta) + r_t \left\{\bar{u}, \left(\frac{2\bar{x}k_x}{R_\Lambda}\right)^{\frac{1}{2}}\bar{v}, \frac{k_x}{k_z}\bar{w}, \frac{k_x}{R_\Lambda}\bar{p}, \bar{\tau}\right\}(\bar{x}, \eta, z, t)$$

- Velocity and temperature similarity solutions:

$$U = \frac{1}{\rho}\frac{\partial \psi}{\partial y} = F'(\eta), \quad V = -\frac{1}{\rho}\frac{\partial \psi}{\partial x} = (2xR_\lambda)^{-1/2}(-TF + \eta_c TF'), \quad T = T(\eta),$$

where

$$\psi = \left(\frac{2x}{R_\lambda}\right)^{1/2} F(\eta), \quad \eta \equiv \left(\frac{R_\lambda}{2x}\right)^{1/2} \int_0^y \rho(x, \check{y})\, d\check{y}, \quad \eta_c = \frac{1}{T}\int_0^\eta T(\check{\eta})\, d(\check{\eta})$$

- In the non linear regime the solution consist of all harmonics:

$$\bar{q} = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \hat{q}_{m,n}(\bar{x}, \eta) e^{imk_x t + ink_z z}, \quad \text{with } \hat{q}_{-m,-n} = (\hat{q}_{m,n})cc$$

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Derivation of the Boundary Region eqs.

- Start from compressible continuity, Navier-Stokes and energy equations.
- Change of variables $(x, y) \mapsto (\bar{x}, \eta)$.
- Use of $k_x \ll 1$, with $k_x R_\lambda = \mathcal{O}(1)$.
- Spanwise diffusivity $\kappa = \mathcal{O}(1)$.
- Nonlinear effects $r_t = \mathcal{O}(1)$.
- Collect nonlinear terms on the right-hand side of the boundary region eqs.

Linear Unsteady Boundary Region eqs. (Ricco & Wu, 2007)

$$\mathcal{L}\left\{\bar{\mathbf{u}}(\bar{x}, \eta), \bar{\tau}(\bar{x}, \eta), \bar{p}(\bar{x}, \eta)\right\} = 0.$$

Nonlinear Unsteady Boundary Region eqs.

$$\mathcal{L}_{m,n}\left\{\hat{\mathbf{u}}(\bar{x}, \eta), \hat{\tau}(\bar{x}, \eta), \hat{p}(\bar{x}, \eta)\right\} = \mathcal{N}_{m,n}.$$

- Continuity

$$\frac{\eta_c}{2\bar{x}}\frac{T'}{T}\hat{u}_{m,n}+\frac{\partial\hat{u}_{m,n}}{\partial\bar{x}}-\frac{\eta_c}{2\bar{x}}\frac{\partial\hat{u}_{m,n}}{\partial\eta}-\frac{T'}{T^2}\hat{v}_{m,n}+\frac{1}{T}\frac{\partial\hat{v}_{m,n}}{\partial\eta}+in\hat{w}_{m,n}$$
$$-\left(\frac{im}{T}+\frac{FT'}{2\bar{x}T^2}\right)\hat{\tau}_{m,n}-\frac{F'}{T}\frac{\partial\hat{\tau}_{m,n}}{\partial\bar{x}}+\frac{F}{2\bar{x}T}\frac{\partial\hat{\tau}_{m,n}}{\partial\eta}=r_tT\hat{C}_{m,n},$$

where

$$\hat{C}_{m,n}=\left[\frac{\partial\widehat{\bar{\rho}\bar{u}}}{\partial\bar{x}}+\frac{\eta_c}{2\bar{x}}\frac{\partial\widehat{\bar{\rho}\bar{u}}}{\partial\eta}-\frac{1}{T}\frac{\partial\widehat{\bar{\rho}\bar{v}}}{\partial\eta}-inT\widehat{\bar{\rho}\bar{w}}+\frac{im}{T}\widehat{\bar{\rho}\bar{\tau}}+\frac{F'}{T}\frac{\partial\widehat{\bar{\rho}\bar{\tau}}}{\partial\bar{x}}-\frac{F}{2\bar{x}T}\frac{\partial\widehat{\bar{\rho}\bar{\tau}}}{\partial\eta}\right]_{m,n},$$

- x-momentum

$$\left(im-\frac{\eta_c}{2\bar{x}}F''+\kappa^2n^2T\mu\right)\hat{u}_{m,n}+F'\frac{\partial\hat{u}_{m,n}}{\partial\bar{x}}-\frac{1}{2\bar{x}}\left(F+\frac{\mu'T'}{T}-\frac{\mu T'}{T^2}\right)\frac{\partial\hat{u}_{m,n}}{\partial\eta}$$
$$-\frac{\mu}{2\bar{x}T}\frac{\partial^2\hat{u}_{m,n}}{\partial\eta^2}+\frac{F''}{T}\hat{v}_{m,n}+\left(\frac{FF''-\mu'F'''-\mu''F''T'}{2\bar{x}T}+\frac{\mu'T'F''}{2\bar{x}T^2}\right)\hat{\tau}_{m,n}$$
$$-\frac{\mu'F''}{2\bar{x}T}\frac{\partial\hat{\tau}_{m,n}}{\partial\eta}=r_tT\hat{X}_{m,n}(\bar{x},\eta)$$

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- y-momentum

$$
\begin{aligned}
&\frac{1}{4\bar{x}^2}\left[FT+\eta_c(FT'-TF')-\eta_c^2F''T\right]\hat{u}_{m,n}+\frac{\mu'T'}{3\bar{x}}\frac{\partial\hat{u}_{m,n}}{\partial\bar{x}}-\frac{\mu}{6\bar{x}}\frac{\partial^2\hat{u}_{m,n}}{\partial\eta\partial\bar{x}}\\
&+\frac{1}{12\bar{x}^2}\left(\mu+\eta_cT'\mu'-\frac{\mu T'\eta_c}{T}\right)\frac{\partial\hat{u}_{m,n}}{\partial\eta}+\frac{\eta_c\mu}{12\bar{x}^2}\frac{\partial^2\hat{u}_{m,n}}{\partial\eta^2}\\
&+\left(im+\frac{\eta_c}{2\bar{x}}F''+\frac{F'}{2\bar{x}}-\frac{FT'}{2\bar{x}T}-\kappa^2n^2\mu T\right)\hat{v}_{m,n}+F'\frac{\partial\hat{v}_{m,n}}{\partial\bar{x}}-\frac{2\mu}{3\bar{x}T}\frac{\partial^2\hat{v}_{m,n}}{\partial\eta^2}\\
&-\frac{1}{\bar{x}}\left(\frac{F}{2}+\frac{2\mu'T'}{3T}-\frac{2\mu T'}{3T^2}\right)\frac{\partial\hat{v}_{m,n}}{\partial\eta}+\frac{\mu'T'}{3\bar{x}}in\hat{w}_{m,n}-\frac{\mu}{6\bar{x}}in\frac{\partial\hat{w}_{m,n}}{\partial\eta}+\frac{1}{2\bar{x}}\frac{\partial\hat{p}_{m,n}}{\partial\eta}\\
&+\frac{1}{4\bar{x}^2}\left[\eta_c\left((FF')'-T\left(\frac{\mu'F''}{T}\right)'\right)-FF'-\frac{F^2T'}{T}-\mu'F''+\frac{4}{3}\left(\frac{\mu'T'F}{T}\right)'\right]\hat{\tau}_{m,n}\\
&-\frac{\mu'F''}{2\bar{x}T}\frac{\partial\hat{\tau}_{m,n}}{\partial\bar{x}}+\left(\frac{\mu'T'F}{3\bar{x}^2T}-\frac{\eta_c\mu'F''}{4\bar{x}^2}\right)\frac{\partial\hat{\tau}_{m,n}}{\partial\eta}=r_tT\hat{Y}_{m,n}(\bar{x},\eta)
\end{aligned}
$$

- z-momentum

$$in\kappa^2 \frac{\eta_c \mu' T' T}{2\bar{x}} \hat{u}_{m,n} - in\kappa^2 \frac{\mu T}{3} \frac{\partial \hat{u}_{m,n}}{\partial \bar{x}} + in\kappa^2 \frac{\eta_c \mu T}{6\bar{x}} \frac{\partial \hat{u}_{m,n}}{\partial \eta} - in\kappa^2 \mu' T' \hat{v}_{m,m} - in\kappa^2 \frac{\mu}{3} \frac{\partial \hat{v}_{m,n}}{\partial \eta}$$
$$+ \left(im + \frac{4n^2\kappa^2 \mu T}{3} \right) \hat{w}_{m,n} + F' \frac{\partial \hat{w}_{m,n}}{\partial \bar{x}} - \frac{1}{2\bar{x}} \left(F + \frac{\mu' T'}{T} - \frac{\mu T'}{T^2} \right) \frac{\partial \hat{w}_{m,n}}{\partial \eta} - \frac{\mu}{2\bar{x} T} \frac{\partial^2 \hat{w}_{m,n}}{\partial \eta^2}$$
$$- \frac{in\kappa^2 F T' \mu'}{3\bar{x}} \hat{\tau}_{m,n} + in\kappa^2 T \hat{p}_{m,n} = r_t T \hat{Z}_{m,n}(\bar{x}, \eta)$$

- Energy

$$-\frac{\eta_c T'}{2\bar{x}} \hat{u}_{m,n} - \frac{M^2(\gamma - 1)\mu F''}{\bar{x} T} \frac{\partial \hat{u}_{m,n}}{\partial \eta} + \frac{T'}{T} \hat{v}_{m,n}$$
$$+ \left[im + \frac{F T'}{2\bar{x} T} - \frac{1}{2\text{Pr}\bar{x}} \left(\frac{\mu' T'}{T} \right)' - \frac{M^2(\gamma - 1)\mu' (F'')^2}{2\bar{x} T} + \frac{\mu n^2 \kappa^2 T}{\text{Pr}} \right] \hat{\tau}_{m,n}$$
$$+ F' \frac{\partial \hat{\tau}_{m,n}}{\partial \bar{x}} - \frac{1}{2\bar{x}} \left(F + \frac{2\mu' T'}{\text{Pr}\, T} - \frac{\mu T'}{\text{Pr}\, T^2} \right) \frac{\partial \hat{\tau}_{m,n}}{\partial \eta} - \frac{\mu}{2\text{Pr}\bar{x} T} \frac{\partial^2 \hat{\tau}_{m,n}}{\partial \eta^2} = r_t T \hat{E}_{m,n}(\bar{x}, \eta)$$

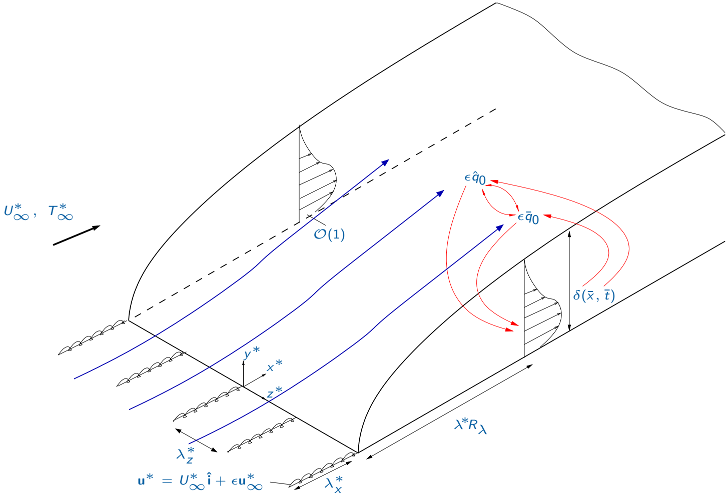
The outer region

- It is crucial to solve the outer-flow dynamics $\rightarrow$ match asymptotic expansion to derive outer boundary conditions for boundary region eqs.
- The displacement effect influences Region IV at leading order.
- Transverse scaling: $\bar{y} = k_x y = \mathcal{O}(1)$.
- Streamwise and temporal scaling: $\bar{x} = k_x x = \mathcal{O}(1)$, $\bar{t} = k_x t = \mathcal{O}(1)$.
- Flow decomposition

$$\begin{Bmatrix} u \\ v \\ w \\ p \\ \tau \\ \rho \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ -1/2 \\ 1 \\ 1 \end{Bmatrix} + \epsilon \begin{Bmatrix} \bar{u}_0 + \hat{u}_0 \\ \bar{v}_0 + \hat{v}_0 \\ \hat{w}_0 \\ \bar{p}_0 \\ \bar{\tau}_0 + \hat{\tau}_0 \\ \bar{\rho}_0 + \hat{\rho}_0 \end{Bmatrix} + \epsilon^2 \begin{Bmatrix} \hat{u}_1 \\ \hat{v}_1 \\ \hat{w}_1 \\ \hat{p}_1 \\ \hat{\tau}_1 \\ \hat{\rho}_1 \end{Bmatrix},$$

where $\bar{q}_0 = \bar{q}_0(\bar{x}, \bar{y}, \bar{t})$ and $\hat{q}_{0,1} = \hat{q}_{0,1}(\bar{x}, y, \bar{y}, z, \bar{t})$.

Outer-flow dynamics



Displacement-induced 2-D disturbance

- Linearized unsteady compressible Euler equations:

$$\frac{\partial \bar{u}_0}{\partial \bar{x}}+\frac{\partial \bar{v}_0}{\partial \bar{y}}+M^2\left(\frac{\partial \bar{p}_0}{\partial \bar{t}}+\frac{\partial \bar{p}_0}{\partial \bar{x}}\right)=0, \quad \frac{\partial \bar{u}_0}{\partial \bar{t}}+\frac{\partial \bar{u}_0}{\partial \bar{x}}+\frac{\partial \bar{p}_0}{\partial \bar{x}}=0, \quad \frac{\partial \bar{v}_0}{\partial \bar{t}}+\frac{\partial \bar{v}_0}{\partial \bar{x}}+\frac{\partial \bar{p}_0}{\partial \bar{y}}=0$$

- Boundary condition

$$\bar{v}_0=\frac{k_x}{\epsilon}\left(\frac{\partial \bar{\delta}}{\partial \bar{x}}+\frac{\partial \bar{\delta}^c}{\partial \bar{t}}\right) \qquad \text{as } \bar{y}\to 0,$$

where $\bar{\delta}$ is the spanwise-averaged boundary-layer displacement thickness

$$\bar{\delta}(\bar{x},\bar{t})=\frac{k_z}{2\pi}\int_0^{2\pi/k_z}\int_0^{\infty}(1-\rho u)\,\mathrm{d}y\mathrm{d}z,$$

and $\bar{\delta}^c$ is an additional thickness due to unsteady and compressible effects

$$\bar{\delta}^c(\bar{x},\bar{t})=\frac{k_z}{2\pi}\int_0^{2\pi/k_z}\int_0^{\infty}(1-\rho)\,\mathrm{d}y\mathrm{d}z.$$

Solution to the 2-D unsteady Euler equations

- Fourier decomposition in time and introduction of the potential

$$\bar{\phi}_0(\bar{x},\bar{y},\bar{t}) = \sum_m \hat{\hat{\phi}}_m(\bar{x},\bar{y}) e^{im\bar{t}}$$

- Continuity can be recast into telegraph equation in terms of $\hat{\hat{\phi}}_m$:

$$\frac{\partial^2 \hat{\hat{\phi}}_m}{\partial \bar{y}^2} + (1-M^2)\frac{\partial^2 \hat{\hat{\phi}}_m}{\partial \bar{x}^2} - 2imM^2 \frac{\partial \hat{\hat{\phi}}_m}{\partial \bar{x}} + m^2 M^2 \hat{\hat{\phi}}_m = 0$$

- Subsonic, transonic or supersonic $\rightarrow$ different solution
  - $M < 1$ Helmholtz equation: $(1-M^2)\frac{\partial^2 \hat{\hat{f}}_m}{\partial \bar{x}^2} + \frac{\partial^2 \hat{\hat{f}}_m}{\partial \bar{y}^2} + \frac{m^2 M^2}{1-M^2}\hat{\hat{f}}_m = 0$
  - $M > 1$: Klein Gordon equation: $(M^2-1)\frac{\partial^2 \hat{\hat{f}}_m}{\partial \bar{x}^2} - \frac{\partial^2 \hat{\hat{f}}_m}{\partial \bar{y}^2} + \frac{m^2 M^2}{M^2-1}\hat{\hat{f}}_m = 0$

where $\hat{\hat{f}}_m(\bar{x},\bar{y}) = \hat{\hat{\phi}}_m(\bar{x},\bar{y}) e^{b/2\bar{x}}$, $\quad b = \frac{-2imM^2}{1-M^2}$.

- Subsonic flow: elliptic case

$$\hat{\hat{\phi}}_m = \frac{1}{2i\sqrt{1-M^2}} e^{\frac{imM^2}{1-M^2}\bar{x}} \int_{\bar{x}-i\bar{y}\sqrt{1-M^2}}^{\bar{x}+i\bar{y}\sqrt{1-M^2}} \frac{\partial \hat{\hat{f}}_m}{\partial \bar{y}}(\xi,0) \cdot J_0\left[\frac{mM}{\sqrt{1-M^2}}\sqrt{\bar{y}^2 + \frac{(\bar{x}-\xi)^2}{1-M^2}}\right] \mathrm{d}\xi$$

where J_0 is the Bessel function of the first kind of zero order.

- Supersonic flow: hyperbolic case

$$\hat{\hat{\phi}}_m = \frac{1}{2\sqrt{M^2-1}} e^{-\frac{imM^2}{M^2-1}\bar{x}} \int_{\bar{x}-\bar{y}\sqrt{M^2-1}}^{\bar{x}+\bar{y}\sqrt{M^2-1}} \frac{\partial \hat{\hat{f}}_m}{\partial \bar{y}}(\xi,0) \cdot I_0\left[\frac{mM}{\sqrt{M^2-1}}\sqrt{\bar{y}^2 - \frac{(\bar{x}-\xi)^2}{M^2-1}}\right] \mathrm{d}\xi$$

where I_0 is the modified Bessel function of the first kind of zero order.

- Transonic flow: cannot be linearised → transonic small perturbation equation (thin-airfoil theory analogy)

$$\frac{\partial^2 \hat{\hat{\phi}}_m}{\partial \bar{y}^2} + \left[1 - M^2 - M^2(\gamma+1)\frac{\partial \hat{\hat{\phi}}_m}{\partial \bar{x}}\right]\frac{\partial^2 \hat{\hat{\phi}}_m}{\partial \bar{x}^2} - 2imM^2\frac{\partial \hat{\hat{\phi}}_m}{\partial \bar{x}} + m^2M^2\hat{\hat{\phi}}_m = 0$$

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Solution to the 3-D part of the unsteady motion

Key: remove the explicit dependance of the 3-D part on the slow varibale $\bar{y}$

- Introduce Prandtl transformation

$$\hat{y} = y - \Re[\tilde{\delta}(\bar{\zeta}, \bar{t})], \quad \bar{\zeta} = \bar{x} + i\bar{y}$$

- Write governing equations for hat-quantities in terms of $\hat{y}$

$$\frac{\partial \hat{v}_0}{\partial \hat{y}} + \frac{\partial \hat{w}_0}{\partial z} = 0.$$

$$\mathcal{L}_{\mathcal{N}} \begin{Bmatrix} \hat{u}_0 \\ \hat{v}_0 \\ \hat{w}_0 \\ \hat{\tau}_0 \end{Bmatrix} = \frac{\epsilon}{k_x} \begin{Bmatrix} 0 \\ -p_{1\hat{y}} \\ -p_{1z} \\ 0 \end{Bmatrix} + \nabla^2 \begin{Bmatrix} \sigma \hat{u}_0 \\ \sigma \hat{v}_0 \\ \sigma \hat{w}_0 \\ \tilde{\sigma} \hat{\tau}_0 \end{Bmatrix},$$

where $\sigma = 1/(k_x R_\lambda)$, $\tilde{\sigma} = \sigma/\mathrm{Pr}$, $\nabla^2 = \partial^2/\partial \hat{y}^2 + \partial^2/\partial z^2$ and $\mathcal{N}$ is the nonlinear differential operator

$$\mathcal{N} = \partial_{\bar{t}} + \partial_{\bar{x}} + \frac{\epsilon}{k_x}\left(\hat{v}_0 \partial_{\hat{y}} + \hat{w}_0 \partial_z\right) - \left[\Re\left(\tilde{\delta}_{\bar{t}} + \tilde{\delta}_{\bar{\zeta}}\right) + \frac{\epsilon}{k_x}\bar{v}_0\right]\partial_{\hat{y}}$$

- Find suitable $\tilde{\delta}$ to satisfy:

$$\Re\left(\tilde{\delta}_{\bar{t}} + \tilde{\delta}_{\bar{\zeta}}\right) = \frac{\epsilon}{k_x}\bar{v}_0 \Rightarrow \tilde{\delta}_{\bar{t}} + \tilde{\delta}_{\bar{x}} = \bar{\delta}_{\bar{x}} + \bar{\delta}^c_{\bar{t}} \quad \text{at } \bar{y} = 0$$

- Leading-order forcing on the boundary layer: $(\hat{v}_0, \hat{w}_0)$.
- A pair of free-stream forcing modes $\rightarrow$ nonlinear terms cancel each other.
- Outer flow solution for $y = \mathcal{O}(1)$

$$(\hat{v}_0, \hat{w}_0, p_1) = \sum_m \sum_n \left(v^\dagger_{m,n}, w^\dagger_{m,n}, p^\dagger_{m,n} \right) e^{im\bar{t} + ink_z z},$$

with the non-null components

$$v^\dagger_{m,\pm 1} = k_z c_\infty e^{-\sigma(k_y^2+k_z^2)\bar{x}} \left[\hat{\chi}_m e^{-i(\bar{x}+k_y y)} + \hat{\chi}^*_{-m} e^{i(\bar{x}+k_y y)} \right],$$

$$w^\dagger_{m,\pm 1} = \pm k_y c_\infty e^{-\sigma(k_y^2+k_z^2)\bar{x}} \left[\hat{\chi}_m e^{-i(\bar{x}+k_y y)} - \hat{\chi}^*_{-m} e^{i(\bar{x}+k_y y)} \right],$$

$$p^\dagger_{0,\pm 2} = 2k_y^2 c_\infty^2 e^{-2\sigma(k_y^2+k_z^2)\bar{x}},$$

$$p^\dagger_{m,0} = -2k_z^2 c_\infty^2 e^{-2\sigma(k_y^2+k_z^2)\bar{x}} \left[\hat{\pi}_m e^{-2i(\bar{x}+k_y y)} + \hat{\pi}^*_{-m} e^{2i(\bar{x}+k_y y)} \right],$$

- Displacement effect $\rightarrow$ temporal harmonics at the edge of the boundary layer.

Boundary-region problem

- Initial conditions: same as Ricco & Wu (2007).
- Boundary condition: matching with the limit as $\bar{y} \to 0$ of the outer flow

$$\hat{u}_{m,n} \to 0, \qquad \hat{\tau}_{m,n} \to 0,$$

$$\hat{v}_{m,n} = \begin{cases} \frac{k_x}{\epsilon}\sqrt{\frac{\sigma}{2\bar{x}}}\left(\frac{\mathrm{d}\hat{\hat{\delta}}_m}{\mathrm{d}\bar{x}} - \frac{\mathrm{d}\delta_{bl}}{\mathrm{d}\bar{x}} + im\hat{\hat{\delta}}_m^c\right) & \text{if } n = 0 \\ \sqrt{\frac{\sigma}{2\bar{x}}}v_{m,n}^{\dagger} & \text{otherwise} \end{cases}$$

$$\hat{w}_{m,n} = k_z\sigma w_{m,n}^{\dagger}, \qquad \hat{p}_{m,n} = \frac{\epsilon}{k_x}p_{m,n}^{\dagger},$$

at $\bar{x} = \mathcal{O}(1)$ and $\eta \to \infty$, where δ_{bl} is the Blasius displacement thickness

$$\delta_{bl} = \int_0^{\infty}\left[1 - \frac{F'(\eta)}{T(\eta)}\right]\mathrm{d}y = \sqrt{\frac{2\bar{x}}{k_x R_\lambda}}(\gamma_c + \beta_c)$$

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Numerical procedure

- Second-order finite difference scheme (central in η, backward in $\bar{x}$).
- Staggering of the pressure.
- Typical Grid: $\Delta\bar{x} = 10^{-4}$, $N_\eta = 2000$ with $\eta_{max} = 60$.
- Number of modes: $N_t = N_z = 9,\ 13$ with $r_t = 2,\ 4$.
- Nonlinear terms $\rightarrow$ pseudo-spectral method.

$$\begin{array}{ccc} \hat{u}_{m,n},\ \hat{v}_{m,n} & \xrightarrow[\text{FFT}]{\text{Inverse}} & \bar{u},\ \bar{v} \\ & & \downarrow \\ \widehat{uv}_{m,n} & \xleftarrow[\text{FFT}]{\text{Forward}} & \bar{u}\bar{v} \end{array}$$

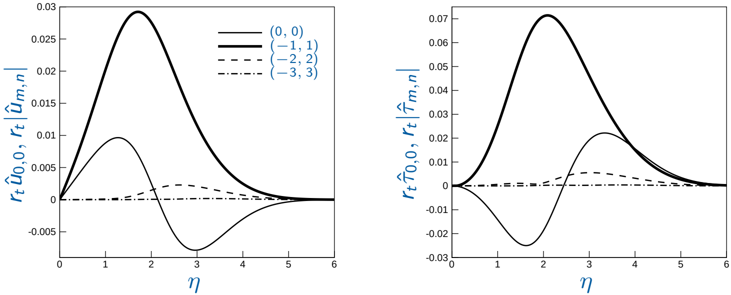
Results

Typical supersonic wind tunnel experiments (no acoustic modes): Graziosi & Brown (2002) (case 1), Fedorov *et al.* (2003) (case 2).

	M_∞	U_∞^* $[\mathrm{ms}^{-1}]$	ν_∞^* $[\mathrm{m}^2\mathrm{s}^{-1}]$ $\times 10^4$	T_∞^* $[K]$	R_λ	k_x $\times 10^3$	κ	$Tu(\%)$	ϵ $\times 10^4$	r_t
Case 1	3	612.0	2.25	103.6	5440	1.63	0.339	0.11	3.89	2.12
Case 2	6	833.4	0.695	49.3	12000	1.20	0.264	0.10	3.53	4.20

Other parameters: $f^* = 500$ Hz and $\lambda_z^* = 0.002$ m (case 1), $f^* = 1000$ Hz, $\lambda_z^* = 0.001$ m (case 2).

- Generation of thermal streaks → secondary instability is altered.
- Significant contribution of the spanwise-uniform mean-flow distortion and of the second harmonic.



Profiles of the streamwise velocity and temperature at $\bar{x} = 1.5$.

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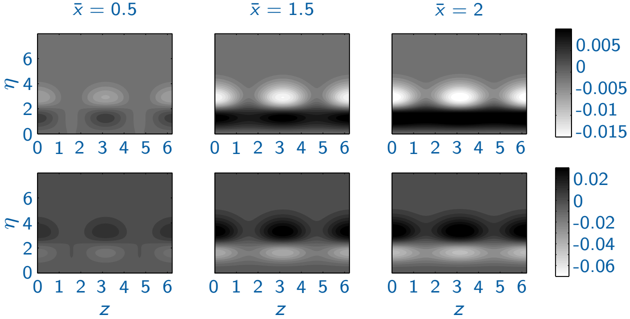
- Time-averaged spanwise modulation of u and τ superimposed on Blasius flow.

$$U_M = F'(\eta) + u_{str}(\bar{x}, \eta, z), \quad T_M = T(\eta) + \tau_{str}(\bar{x}, \eta, z), \quad q_{str} = \sum_{n=-(N_z-1)/2}^{(N_z-1)/2} \hat{q}_{0,n} e^{ink_z z}$$

- Flow is $\begin{cases} \text{accelerated and cooled close to the wall} \\ \text{decelerated and heated close to the free stream.} \end{cases}$



Contour of u_{str} (first row) and τ_{str} (second row) in $\eta - z$ plane.

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Effect of Mach number

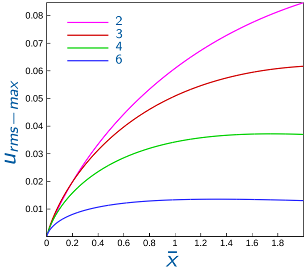
M_∞	U^*_∞ [ms^{-1}]	ν^*_∞ [m^2s^{-1}] $\times 10^4$	T^*_∞ [K]	R_λ	k_x $\times 10^{-3}$	κ	$Tu[\%]$	ϵ $\times 10^{-4}$	r_t
2	508.9	1.87	161.1	5440	1.97	0.305	0.11	3.89	2.12
3	612.0	2.25	103.6	5440	1.63	0.339	0.11	3.89	2.12
4	666.3	2.44	69.0	5440	1.50	0.350	0.11	3.89	2.12
6	715.3	2.63	35.4	5440	1.39	0.363	0.11	3.89	2.12

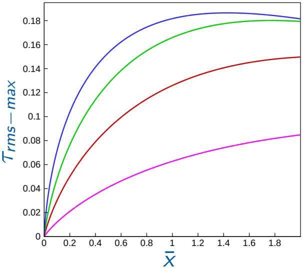
Effect of Mach number

- Maximum of the r.m.s. of the velocity and temperature fluctuations

$$q_{rms-max} = \max_{\eta} \left\{ r_t \left(\sum_{m=-(N_t-1)/2}^{(N_t-1)/2} \sum_{n=-(N_z-1)/2}^{(N_z-1)/2} |\hat{q}_{m,n}|^2 \right)^{1/2} \right\}, \quad m \neq 0$$

- Stabilizing effect on streamwise velocity, enhancing effect on temperature.





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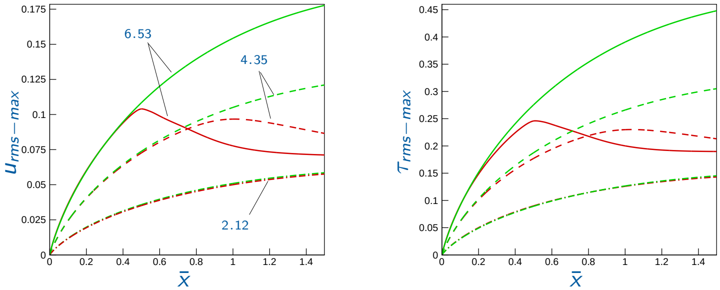
Effect of FST intensity

M_∞	U^*_∞ [ms^{-1}]	ν^*_∞ [m^2s^{-1}] $\times 10^4$	T^*_∞ [K]	R_λ	k_x $\times 10^{-3}$	κ	Tu[%]	ϵ $\times 10^{-4}$	r_t
3	612.0	2.25	103.6	5440	1.63	0.339	0.11	3.89	2.12
3	612.0	2.25	103.6	5440	1.63	0.339	0.23	8.0	4.35
3	612.0	2.25	103.6	5440	1.63	0.339	0.34	12.0	6.53

Effect of FST intensity

- Overlapping of linear (green) and nonlinear (red) solutions at small $\bar{x}$.
- Stabilizing effect on both the streamwise velocity and temperature.
- Abrupt deviation of the nonlinear trend from the linear solution at $\bar{x} = 0.5$.



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Conclusions

- **Boundary-region approach** → nonlinear evolution of unsteady streaks in a compressible boundary layer.
- Abrupt departure of the nonlinear solution from the linear one.
- Relevance of **nonlinear effects** even with small turbulence level → 'quiet' environments such as free flight.

Future work

- **Secondary instability analysis**.
- Flow control: different **control methods** for high speed flows (blowing and suction, cooling, *etc.* ...).

Thank you

